<u>Background knowledge</u> - <u>Differential calculus</u>

A1 <u>Limits</u>:

E.g.1 Determine: (a) $\lim_{x \to -1}(2x + 1)$ = *[2(-1) + 1] =* ***-1*** *Substitute x with -1*

(b) $\lim_{x \to 3}\left(\frac{x^2 - 9}{x - 3}\right)$ *Cannot substitute x with 3 yet,*

$= \lim_{x \to 3}\left(\frac{(x - 3)(x + 3)}{(x - 3)}\right)$ *because (3 – 3 = 0) and denominator ≠ 0.*

$= \lim_{x \to 3}\left(\frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}}\right)$ *∴ First factorise and then simplify*

$= \lim_{x \to 3}(x + 3) = 3 + 3 = \mathbf{6}$

A2 <u>Derivatives from first principles</u>:

Formula for first principles:

$$f'(x) = \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$$

E.g.2 Find $f'(x)$ *if* $f(x) = 2x^2 - x$

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$f(x) = 2x^2 - x$

$\therefore\ f(x+h) = 2(x+h)^2 - (x+h)$

$\therefore\ f(x+h) = 2(x^2 + 2xh + h^2) - (x+h)$

$\therefore\ f(x+h) = 2x^2 + 4xh + 2h^2 - x - h$

But $f'(x) = \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$

$f'(x) = \lim_{h \to 0}\frac{(2x^2 + 4xh + 2h^2 - x - h) - (2x^2 - x)}{h}$

$= \lim_{h \to 0}\frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{x} - h - \cancel{2x^2} + \cancel{x}}{h}$

$= \lim_{h \to 0}\frac{4xh + 2h^2 - h}{h}$

$= \lim_{h \to 0}\frac{\cancel{h}(4x + 2h - 1)}{\cancel{h}}$

$= \lim_{h \to 0}(4x + 2h - 1)$

$= 4x + 2(0) - 1 = \mathbf{4x - 1}$

A3 Derivative rules:
A3.1 Rules:

* Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

* Derivative of a constant rule: $\frac{d}{dx}(k) = 0$ for k as any constant value

* Derivative of the sum or difference of two functions:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

* Derivative of a constant multiplied with a function:

$$\frac{d}{dx}[k \times f(x)] = k\frac{d}{dx}[f(x)]$$ for k as any constant value

A3.2 Notations:

E.g.3 (a) Find $f'(x)$ *if* $f(x) = \frac{5}{x} - 3x + 2x^5 - 1$

$$\therefore f(x) = 5x^{-1} - 3x + 2x^5 - 1$$

$$\therefore f'(x) = -5x^{-2} - 3 + 10x^4$$

$$\therefore \boldsymbol{f'(x) = \frac{-5}{x^2} - 3 + 10x^4}$$

(b) Find $D_x[(3x + m)^2]$

$$= D_x[3x^2 + 6mx + m^2]$$

$$= \boldsymbol{6x + 6m}$$ *[m is seen as a constant value]*

(c) Find $\frac{dy}{dx}$ *if* $y = \sqrt[3]{8x^2} + \frac{x^2 - 2x - 8}{x - 4}$

$$\therefore y = \sqrt[3]{8} \times \sqrt[3]{x^2} + \frac{(x-4)(x+2)}{(x-4)}$$

$$\therefore y = 2 \times x^{\frac{2}{3}} + \frac{(x-4)(x+2)}{(x-4)}$$

$$\therefore y = 2x^{\frac{2}{3}} + \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}}$$

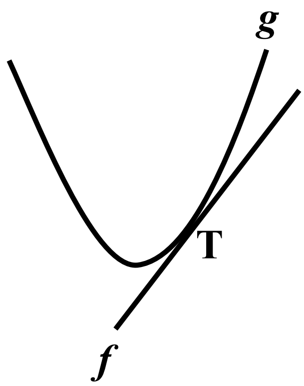
$$\therefore y = 2x^{\frac{2}{3}} + x + 2$$

$$\therefore \frac{dy}{dx} = 2 \times \frac{2}{3}x^{\frac{2}{3}-1} + 1$$

$$\therefore \boldsymbol{\frac{dy}{dx} = \frac{4}{3}x^{\frac{-1}{3}} + 1}$$

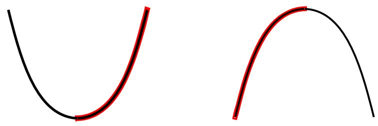
A4 Applications:

A4.1 Basic application:

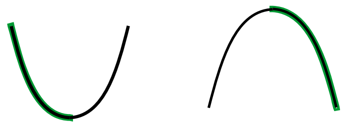


* To find the equation of a tangent to any curve:

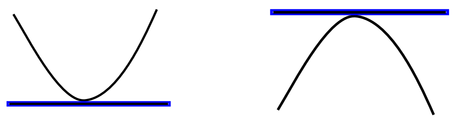
  f(x) = $y - y_1 = m(x - x_1)$
  with T (x_1 ; y_1) the common point
  m = $g'(x_1)$.



* A function is increasing if gradient is positive.



* A function is decreasing if gradient is negative.



* A function is neither increasing or decreasing
  if the gradient is 0.

E.g.4 Find the coordinates of S, if S is the point of tangency between the tangent $y = 3x - 7$ and the curve $f(x) = x^2 + x - 6$.

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Tangent: $y = 3x - 7 \quad \rightarrow \quad m = 3$

But $\quad m = f'(x) = 2x + 1$

$\therefore \quad 3 = 2x + 1$

$\therefore \quad 3 = 2x + 1$

$\therefore \quad 2 = 2x$

$\therefore \quad x = 1$

Substitute $x = 1$ into the tangent or into f(x):

$y = 3x - 7$ *or* $f(x) = x^2 + x - 6$
$y = 3(1) - 7$ $\quad y = (1)^2 + (1) - 6$
$y = -4$ $\quad y = 1 + 1 - 6 = -4$

$\therefore$ ***S (1 ; -4)***

A4.2 Graphs:
A4.2.1 Solving cubic equations:

$\therefore$ Different examples of finding the x-intercept: (y = 0)

(1) If $f(x) = x^3 + 2x^2 + x$

x-intercepts: ($y = 0$) - [Common factor x .]

$\therefore\ 0 = x^3 + 2x^2 + x$

$\therefore\ 0 = x(x^2 + 2x + 1)$

$\therefore\ 0 = x(x + 1)(x + 1)$

$\therefore\ \boldsymbol{x = 0}$ or $\boldsymbol{x = -1}$ or $\boldsymbol{x = -1}$

(2) If $f(x) = (1 - 2x)(x + 3)(x + 2)$

x- intercepts: ($y = 0$) - [Factors given.]

$\therefore\ 0 = (1 - 2x)(x + 3)(x + 2)$

$\therefore\ 1 - 2x = 0$ or $x + 3 = 0$ ofr $x + 2 = 0$

$\therefore\ \boldsymbol{x = \frac{1}{2}}$ or $\boldsymbol{x = -3}$ or $\boldsymbol{x = -2}$

(3) If $f(x) = x^3 + 2x^2 - x - 2$

x- intercepts: ($y = 0$) - [Factorise by grouping.]

$\therefore\ 0 = x^3 + 2x^2 - x - 2$

$\therefore\ 0 = x^2(x + 2) - 1(x + 2)$

$\therefore\ 0 = (x + 2)\,(x^2 - 1)$

$\therefore\ 0 = (x + 2)\,(x - 1)(x + 1)$

$\therefore\ \boldsymbol{x = -2}$ or $\boldsymbol{x = 1}$ or $\boldsymbol{x = -1}$

(4) If $f(x) = -4x^3 + 28x^2 - 49x + 18$

<u>x- intercepts</u>: ($y = 0$) - [Factor theorem and synthetic division.]

$\therefore\ f(1) = -4(1)^3 + 28(1)^2 - 49(1) + 18 = 91$

$\therefore\ x - 1$ is not a factor of $f(x)$.

$\therefore\ f(2) = -4(2)^3 + 28(2)^2 - 49(2) + 18 = 0$

$\therefore\ (x - 2)$ is a factor of $f(x)$ $\longrightarrow$ $x = 2$ is the one x- intercept

	-4	+ 28	- 49	+ 18	$\longrightarrow$	coefficients of $f(x)$
2	0	-8	+ 40	- 18		
	- 4	+ 20	- 9	0	$\longrightarrow$	coefficients of ( )

Always 0

$\therefore\ 0 = -4x^3 + 28x^2 - 49x + 18$

$\therefore\ 0 = (x - 2)(-4x^2 + 20x - 9)$

$\therefore\ x - 2 = 0$ or $-4x^2 + 20x - 9 = 0$

$\therefore\ \boldsymbol{x = 2}$ or $4x^2 - 20x + 9 = 0$

$\therefore\ (2x - 1)(2x - 9) = 0$

$\therefore\ \boldsymbol{x = \frac{1}{2}}$ or $\boldsymbol{x = \frac{9}{2}}$

(5) If $f(x) = x^3 - 5x^2 + 7x - 3$

<u>x- intercepts</u>: ($y = 0$) - [Factor theorem and inspection.]

$\therefore\ 0 = x^3 - 5x^2 + 7x - 3$

$\therefore\ f(1) = (1)^3 - 5(1)^2 + 7(1) - 3 = 0$

$\therefore\ (x - 1)$ is a factor of $f(x)$

$\therefore\ x^3 - 5x^2 + 7x - 3 = (x - 1)(x^2 \ldots\ldots\ldots\ldots\ ?)$

$\therefore\ x^3 - 5x^2 + 7x - 3 = (x - 1)(x^2 \pm ?x + 3)$ because $(-1)(+3) = -3$

$\therefore\ x^3 - 5x^2 + 7x - 3 = x^3 \ldots\ldots + 3x - 1x^2 \ldots\ldots - 3$

$\therefore\ x^3 - 5x^2 + 7x - 3 = x^3 + 4x + 3x - 1x^2 - 4x^2 - 3$

$\therefore\ x^3 - 5x^2 + 7x - 3 = (x - 1)(x^2 - 4x + 3)$

$\therefore\ f(x) = (x - 1)(x^2 - 4x + 3) = (x - 1)(x - 3)(x - 1) = 0$

$\therefore\ \boldsymbol{x = 1}$ or $\boldsymbol{x = 3}$ or $\boldsymbol{x = 1}$

(6) If $f(x) = 2x^3 - 2x^2 - 13x - 9$

<u>x- intercepts</u>: ($y = 0$) - [Factor theorem and synthetic division – formula.]

$\therefore\ f(1) = 2(1)^3 - 2(1)^2 - 13(1) - 9 = -22$

$\therefore\ (x - 1)$ is not a factor of $f(x)$.

$\therefore\ f(-1) = 2(-1)^3 - 2(-1)^2 - 13(-1) - 9 = 0$

$\therefore\ (x + 1)$ is a factor of $f(x)$

$$\begin{array}{r|rrrr} & 2 & -2 & -13 & -9 \\ -1 & 0 & -2 & +4 & +9 \\ \hline & 2 & -4 & -9 & +0 \end{array}$$

$\therefore\ (x + 1)(2x^2 - 4x - 9) = 0$

$\therefore\ (x + 1) = 0$ or $(2x^2 - 4x - 9) = 0$

$\therefore\ \boldsymbol{x = -1}$ $\qquad\therefore\ a = 2\ ;\ b = -4$ *and* $c = -9$

$$\therefore\quad x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-9)\,ac}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16 + 72}}{4}$$

$$= \frac{4 \pm \sqrt{88}}{4}$$

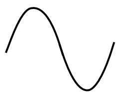
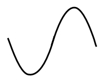
$$x = \frac{4}{4} \pm \frac{2\sqrt{22}}{4}$$

$$\therefore\ \boldsymbol{x = 1 \pm \frac{\sqrt{22}}{2}}$$

A4.2.2 <u>Sketching of cubic graphs</u>:

Standard form: $y = ax^3 + bx^2 + cx + d$

To sketch the graph, determine:

* **Form**: if $a > 0 \rightarrow$  or if $a < 0 \rightarrow$ 

* ***y*- intercept** $\rightarrow$ $x = 0$ $\therefore$ $(0\,;\,d)$

* ***x*- intercept(s)** $\rightarrow$ $y = 0$ [See A4.2.1]

* **Stationary points (turning points)** $\rightarrow$ $f'(x) = 0$
Also the local minimum and local maximum.

* **Point of infection** $\rightarrow$ $f''(x) = 0$

E.g.5 Sketch: $f(x) = x^3 + x^2 - x - 1$

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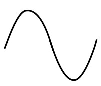
* **Form:** if $a > 0 \rightarrow$ 

* ***y*- intercept** $\rightarrow$ $x = 0$

$f(0) = (0)^3 + (0)^2 - (0) - 1 = -1$

$\therefore$ $(0\,;\,-1)$

* ***x*- intercept(s)** $\rightarrow$ $y = 0$

$\therefore$ $0 = x^3 + 1x^2 - x - 1$

$\therefore$ $0 = x^2(x + 1) - 1(x + 1)$

$\therefore$ $0 = (x + 1)(x^2 - 1)$

$\therefore$ $0 = (x + 1)(x - 1)(x + 1)$

$\therefore$ $\boldsymbol{x = -1}$ or $\boldsymbol{x = 1}$ or $\boldsymbol{x = -1}$

$\therefore$ $(-1\,;\,0)$; $(1\,;\,0)$ and $(-1\,;\,0)$ $\rightarrow$ If 2 x- intercepts are the same, it means that the graph will turn on the x-axis. $\therefore$ $(-1\,;\,0)$ is then also a turning point.

* **Stationary points(Turning point)** $\rightarrow f'(x) = 0$

$$f(x) = x^3 + x^2 - x - 1$$

$$f'(x) = 3x^2 + 2x - 1$$

$$0 = (x + 1)(3x - 1)$$

x- intercept (-1 ; 0) $\rightarrow$ turning point.

$\therefore x = -1$ or $x = \frac{1}{3}$

$\therefore f(-1) = (-1)^3 + (-1)^2 - (-1) - 1$

$\therefore f(-1) = 0$

$\therefore (-1\,; 0)$

$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) - 1$

$f\left(\frac{1}{3}\right) = \frac{-32}{27}$

$\left(\frac{1}{3}\,; \frac{-32}{27}\right)$

* **Point of inflection** $\rightarrow f''(x) = 0$

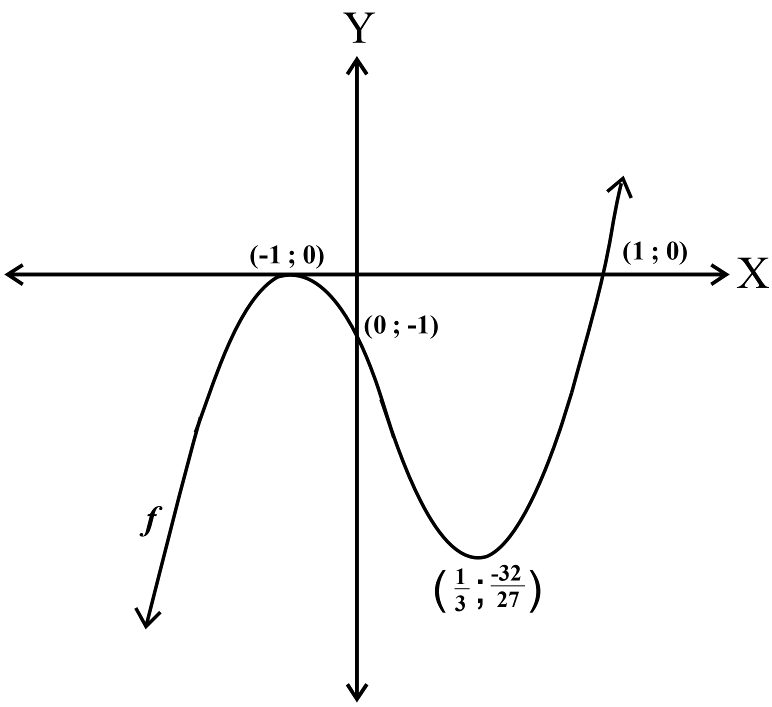
$$f(x) = x^3 + x^2 - x - 1$$

$$f'(x) = 3x^2 + 2x - 1$$

$$f''(x) = 6x + 2 = 0$$

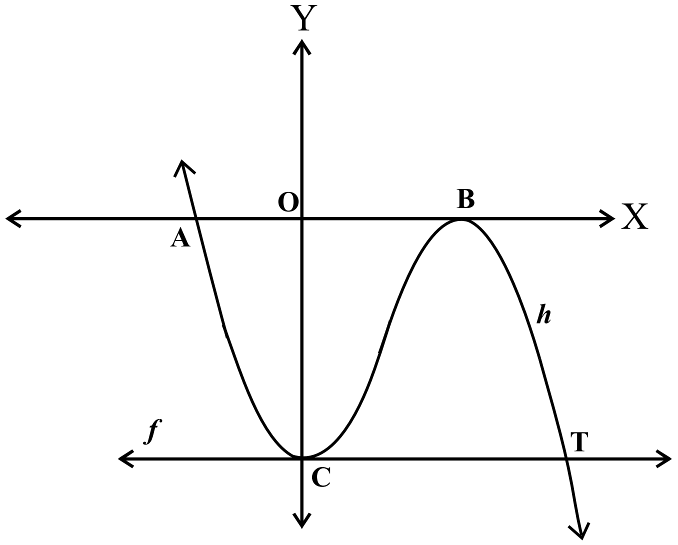
$$\therefore x = \frac{-2}{6} = \frac{-1}{3}$$

SKETCH:



A4.2.2 How to find the equation of a cubic graph:

E.g.6 $h(x) = ax^3 + bx^2 + cx + d$ *with A(-1 ; 0) ; B(2 ; 0) and C(0 ; -8)*
CT is parallel to the x-axis.



Calculate: (a) the values of a ; b ; c and d.
(b) the equation of f.
(c) the equation of the tangent (g) to curve h, through A.
(d) the x-coordinate of the point of inflection of h.
(e) the coordinates of T.
(f) the average gradient between B and C.
(g) the x-values for which h will be decreasing.
(h) x if h(x) < 0.

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(a) $x = -1$ or $x = 2$ or $x = 2$ *coordinates of A and B*

$\therefore h(x) = a(x + 1)(x - 2)(x - 2)$ *curve turning at B → x = 2*

$\therefore h(x) = a(x + 1)(x^2 - 4x + 4)$

$\therefore h(x) = a(x^3 - 4x^2 + 4x + 1x^2 - 4x + 4)$

$\therefore h(x) = a(x^3 - 3x^2 + 4)$

x y → y = h(x)

$\therefore h(0) = a[(0)^3 - 3(0)^2 + 4]$ *through C(0 ; -8)*

$\therefore -8 = a(4)$

$\therefore a = -2$

$\therefore h(x) = -2(x^3 - 3x^2 + 4)$

$\therefore h(x) = -2x^3 + 6x^2 - 8$

$\therefore$ **$a = -2$; $b = +6$; $c = 0$ and $d = -8$**

(b) $\boldsymbol{f(x) = -8}$ tangent to TP $\rightarrow m = 0$

(c) Equation of tangent : $y - y_1 = m(x - x_1)$ with $m = h'(x)$

$\therefore \; h'(x) = -6x^2 + 12x$

$\therefore \; h'(-1) = -6(-1)^2 + 12(-1)$ at $A \rightarrow x = -1$

$\therefore \; m = -18$

$\therefore \; y - 0 = -18(x - (-1))$ through $A(-1 ; 0)$

$\therefore \; \boldsymbol{y = -18x - 18}$

(d) Point of inflection $\rightarrow h''(x) = 0$

$h(x) = -2x^3 + 6x^2 - 8$

$\therefore \; h'(x) = -6x^2 + 12x$

$\therefore \; h''(x) = -12x + 12 = 0$

$-12x = -12$

$\therefore \; \boldsymbol{x = 1}$

(e) T is the point of intersection of h and f:

$h(x) = -2x^3 + 6x^2 - 8$ and $f(x) = -8 = y \rightarrow$ substitute into h

$\therefore \; -8 = -2x^3 + 6x^2 - 8$

$\therefore \; 0 = -2x^3 + 6x^2$

$\therefore \; 0 = x^3 - 3x^2$

$\therefore \; 0 = x^2(x - 3)$

$\therefore \; x = 0$ or $x = 3$

$\therefore \; \boldsymbol{T(3 ; -8)}$

(f) $AG = \dfrac{y_2 - y_1}{x_2 - x_1}$ through $\overset{x_1 \;\; y_1}{B(2 ; 0)}$ and $\overset{x_2 \;\; y_2}{C(0 ; -8)}$

$\therefore \; AG = \dfrac{-8 - 0}{0 - 2} = \dfrac{-8}{-2} = \boldsymbol{4}$

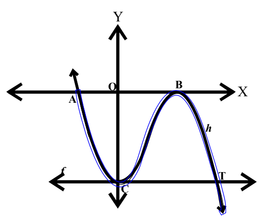
(g) h is decreasing if $x \in (-\infty ; \mathbf{0}) \cup (\mathbf{2} ; \infty)$

(h) $h(x) < 0$

$\boldsymbol{x > -1 \text{ but } x \neq 2}$

$\uparrow$

$h(x) < 0$ and not $h(x) \leq 0$

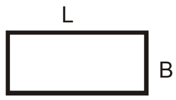


A4.3 Further applications:

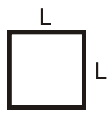
- Minima / maxima $\rightarrow \quad f'(x) = 0$
- Distance = speed $\times$ time $\rightarrow$ Speed / velocity is the rate of change of the distance travelled over a period of time.
  $\therefore$ If s(t) the distance travelled, after a period of time t, then the velocity will be $\rightarrow s'(t)$
- Acceleration is the tempo of change of velocity $\rightarrow \therefore s''(t)$
- Any rate of change: $f'(x)$
- Maximum / minimum perimeter, area or volume $\rightarrow f'(x) = 0$

Revise the formula for perimeter, area or volume:

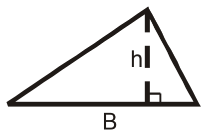
* Rectangle: Perimeter $= 2L + 2B$
  Area $= L \times B$



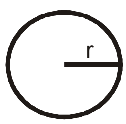
* Square: Perimeter $= 4L$
  Area $= L^2$



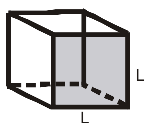
* Triangle: Perimeter $=$ sy + sy + sy
  Area $= \frac{1}{2} B \times \perp h$



* Circle: Circumference $= 2\pi r$
  Area $= \pi r^2$

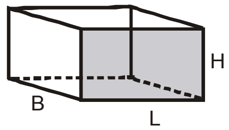


* Cube: $\therefore$ **Surface area = $6L^2$**
  and **Volume = L^3**



* Rectangular prism: $\therefore$ **Surface area = 2LH + 2BH + 2LB**

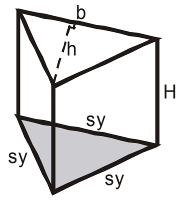
and **Volume = L × B × H**



* Triangular prism: **Surface area = (side + side + side)H + $2 \times \frac{1}{2}$ bh**

$\therefore$ **Surface area = (side + side + side)H + bh**

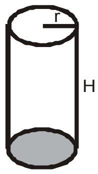
and **Volume = $\frac{1}{2}$bh × H**



* Cylinder: **Surface area = $(2\pi r)H + 2(\pi r^2)$**

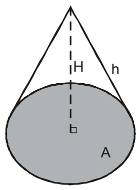
$\therefore$ **Surface area = $2\pi r\,(H + r)$**

and **Volume = V = $\pi r^2 H$**



* Real cone:

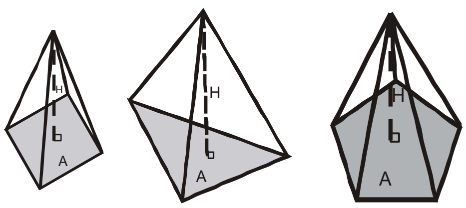
Surface area = $\pi r(h + r)$ **With: A the area of the base – needed for the volume!**
r the radius of the base (circle)
h the slant height of the cone
H the perpendicular height – needed for the volume!



and **Volume = $\frac{1}{3}$AH**

* Pyramid:

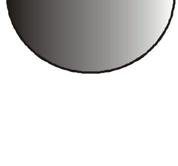
Surface area = A + $\frac{1}{2}$ ph **With: A the area of the base – needed for the volume!**
r the radius of the base
h the slant height of the pyramid
H the perpendicular height – needed for the volume!



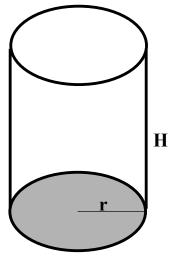
and **Volume = $\frac{1}{3}$AH**

* Sphere: **Surface area = $4\pi r^2$ with r the radius.**

and **Volume** = $\frac{4}{3}\pi r^3$



E.g.7 *An open cylindrical cardboard container is manufactured.*
The volume of the container is 2 155 cm^3.



(a) *Write* H *in terms of the volume and* r.

(b) *Show that the total surface area of the cylinder is:* $P(r) = \pi r^2 + \frac{4\,310}{r}$

(c) *Calculate* r, *correct to 2 decimals, which gives minimum surface area.*

****

(a) $V = \pi r^2 H$

$\therefore\ 2\,155 = \pi r^2 H$

$\therefore\ \boldsymbol{H = \frac{2\,155}{\pi r^2}}$

(b) $SA = 2\pi r^2 + 2\pi r H - \pi r^2$ *Cylinder is open on one side – only one circle!*

$\therefore\ P(r) = \pi r^2 + 2\cancel{\pi r}\left(\frac{2\,155}{\cancel{\pi r^2}}\right)$ *Substitue* H *as determined in (a).*

$= \pi r^2 + 2\left(\frac{2\,155}{r}\right)$

$\therefore\ \boldsymbol{P(r) = \pi r^2 + \frac{4\,310}{r}}$

(c) *Minimum* $\rightarrow P'(r) = 0$

But $P(r) = \pi r^2 + \frac{4\,310}{r} = \pi r^2 + 4\,310 r^{-1}$

$\therefore\ P'(r) = 2\pi r - 4\,130 r^{-2} = 0$

$\therefore\ 2\pi r = \frac{4\,130}{r^2}$

$\therefore\ 2\pi r^3 = 4\,130$

$\therefore\ r^3 = \frac{4\,130}{2\pi}$

$\therefore\ r = \sqrt[3]{657{,}309915}$

$\therefore\ \boldsymbol{r = 8{,}69\ cm}$

E.g.8 *The rate at which organism "X" grows is given by:*
$g(t) = -t^2 + 10t + 62$ *with* t *the period of time in minutes and the mass of the organism* g *in gram.*

Determine: *(a) the initial mass of organism "X".*
(b) the mass of organism "X" after 3 minutes.
(c) after how many minutes organism "X" will reach maximum mass.
(d) the maximum mass of organism "X".

****

(a) Initial $\rightarrow t = 0$

$\therefore\ g(0) = -(0)^2 + 10(0) + 62 = 62$

$\therefore$ *Initial mass of organism "X" is 62 grams.*

(b) After 3 minutes $\rightarrow t = 3$

$\therefore\ g(3) = -(3)^2 + 10(3) + 62 = 83$

$\therefore$ *After 3 minutes the mass of organism "X" is 83 grams.*

(c) Maximum mass $\rightarrow g'(t) = 0$

$$g(t) = -t^2 + 10t + 62$$

$$\therefore\ g'(t) = -2t + 10 = 0$$

$$\therefore\ -2t = -10$$

$$\therefore\ t = 5$$

$\therefore$ *Organism "X" reaches maximum mass after 5 minutes.*

(d) Maximum mass after 5 minutes.

$\therefore\ g(5) = -(5)^2 + 10(5) + 62 = 87$

$\therefore$ *Organism "X" has a maximum mass of 87 grams.*